Ex 1: Knapsack Problem

- Knapsack has finite volume, $K$
- Can fill knapsack with integer number of items, $x_i$
- Each item has per unit volume of $v_i$
- Each item has per unit value of $c_i$

**Goal:** Select number of items $x_i$ to place in knapsack to max total value.
Let $V(y)$ represent the maximal knapsack value if the remaining volume is $y$.

Consider one unit of item $i$:
- value added, $c_i$
- volume remaining, $y - v_i$
- maximal value of knapsack with remaining volume $V(y - v_i)$

Principle of Optimality & Boundary Condition:

\[
V(y) = \max_{v_i \leq y} \{ c_i + V(y - v_i) \}
\]

$V(0) = 0$
Chris McCandless is traveling into the wilderness. He can bring one knapsack including food and equipment. The knapsack has a finite volume. However, he wishes to maximize the total “value” of goods in the knapsack.

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s. to} & \quad 2x_1 + 3x_2 \leq 9 \\
& \quad x_i \geq 0 \in \mathbb{Z}
\end{align*}
\]

\[
\begin{align*}
V(0) & = 0 \\
V(1) & = \max_{v_i \leq 1} \{ c_i + V(1 - v_i) \} = 0 \\
V(2) & = \max_{v_i \leq 2} \{ c_i + V(2 - v_i) \} \\
& = 2 + V(2 - 2) = 2 \\
V(3) & = \max_{v_i \leq 3} \{ c_i + V(3 - v_i) \} \\
& = \max \{ 2 + V(2 - 2), 1 + V(3 - 3) \} = 2 \\
V(4) & = \max_{v_i \leq 4} \{ c_i + V(4 - v_i) \} \\
& = \max \{ 2 + V(4 - 2), 1 + V(4 - 3) \} = \max \{ 2 + 2, 1 + 0 \} = 4
\end{align*}
\]
From the Midterm cont.

\[ V(5) = \max_{v_i \leq 5} \{ c_i + V(5 - v_i) \} \]
\[ = \max \{ 2 + V(5 - 2), 1 + V(5 - 3) \} = \max \{ 2 + 2, 1 + 2 \} = 4 \]

\[ V(6) = \max_{v_i \leq 6} \{ c_i + V(6 - v_i) \} \]
\[ = \max \{ 2 + V(6 - 2), 1 + V(6 - 3) \} = \max \{ 2 + 4, 1 + 2 \} = 6 \]

\[ V(7) = \max_{v_i \leq 7} \{ c_i + V(7 - v_i) \} \]
\[ = \max \{ 2 + V(7 - 2), 1 + V(7 - 3) \} = \max \{ 2 + 4, 1 + 4 \} = 6 \]

\[ V(8) = \max_{v_i \leq 8} \{ c_i + V(8 - v_i) \} \]
\[ = \max \{ 2 + V(8 - 2), 1 + V(8 - 3) \} = \max \{ 2 + 6, 1 + 4 \} = 8 \]

\[ V(9) = \max_{v_i \leq 9} \{ c_i + V(9 - v_i) \} \]
\[ = \max \{ 2 + V(9 - 2), 1 + V(9 - 3) \} = \max \{ 2 + 6, 1 + 6 \} = 8 \]

Item 1 - food: 4 units
Item 2 - equipment: 0 units
Ex 2: Smart Appliance Scheduling

- appliance (say, dishwasher) has five cycles (each 15 min)

<table>
<thead>
<tr>
<th>cycle</th>
<th>power</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>prewash  1.5 kW</td>
</tr>
<tr>
<td>2</td>
<td>main wash 2.0 kW</td>
</tr>
<tr>
<td>3</td>
<td>rinse 1    0.5 kW</td>
</tr>
<tr>
<td>4</td>
<td>rinse 2    0.5 kW</td>
</tr>
<tr>
<td>5</td>
<td>dry       1.0 kW</td>
</tr>
</tbody>
</table>

- cycle must be run in order, possibly with idle periods in between

- electricity price varies (in 15 min periods)

- find cheapest cycle schedule starting at 17:00 and ending at 24:00
Electricity Price

Time of Day

Electricity Cost [cents/kWh]

00:00 04:00 08:00 12:00 16:00 20:00 24:00

0 5 10 15 20 25 30 35

0:00:00 04:00 08:00 12:00 16:00 20:00 24:00

Time of Day

Electricity Cost [cents/kWh]
Formulation

- $k$ indexes 15 min periods; $k = 0$ is 17:00–17:15, $k = 28$ is 24:00–24:15
- $x_k \in \{0, \cdots, 5\}$ is the current cycle; $x_0 = 0$.
- $u_k \in \{0, 1\}$ corresponding to (wait, next cycle).
- state-transition function: $x_{k+1} = f(x_k, u_k) = x_k + u_k$
- cost-per-time-step: $c_k(x_k, u_k) = \frac{1}{4} c_k p_{x_{k+1}} u_k$
  - $c_k$ is the electricity cost in cents/kWh in period $k$
  - $p_i$ is power of cycle $i$
- terminal cost: $c_N(x_N) = 0$ for $x_N = 5$; $c_N(x_N) = \infty$ otherwise.
Let $V_k(x_k)$ represent min cost-to-go from time step $k$ to $N$, given current state $x_k$.

Principle of Optimality:

$$V_k(x_k) = \min_{u_k \in \{0, 1\}} \left\{ \frac{1}{4} c_k p_{x_{k+1}} u_k + V_{k+1}(x_{k+1}) \right\}$$

$$= \min_{u_k \in \{0, 1\}} \left\{ \frac{1}{4} c_k p_{x_{k+1}} u_k + V_{k+1}(x_k + u_k) \right\}$$

$$= \min \left\{ V_{k+1}(x_k), \frac{1}{4} c_k p_{x_{k+1}} + V_{k+1}(x_k + 1) \right\}$$

with the boundary condition

$$V_N(5) = 0, \quad V_N(i) = \infty \quad \text{for} \quad i \neq 5$$

Optimal Control Action:

$$u_k^*(x_k) = \arg \min_{u_k \in \{0, 1\}} \left\{ \frac{1}{4} c_k p_{x_{k+1}} u_k + V_{k+1}(x_{k+1}) \right\}$$
%% Problem Data
% Cycle power
p = [0; 1.5; 2.0; 0.5; 0.5; 1.0];

% Electricity Price Data
c = ... 
[12,12,12,10,9,8,8,7,7,6,5,5,5,5,5,5,6,7,7,8,9,9,10,11,11,...
12,12,14,15,15,16,17,19,19,20,21,21,22,22,22,20,20,19,17,15,15,16,
17,17,18,18,16,16,17,17,18,20,20,21,21,21,20,20,19,19,18,17,17, ...
16,19,21,22,23,24,26,26,27,28,30,30,30,29,28,28,26,23,21,20,18,

%% Solve DP Equations
% Time Horizon
N = 28;
% Number of states
nx = 6;

% Preallocate Value Function
V = inf*ones(N,nx);
% Preallocate control policy
u = nan*ones(N,nx);
% Boundary Condition
V(end,end) = 0;
% Iterate through time backwards
for k = (N-1):-1:1;

    % Iterate through states
    for i = 1:nx

        % If you're in last state, can only wait
        if(i == nx)
            V(k,i) = V(k+1,i);

        % Otherwise, solve Principle of Optimality
        else

            % Choose u=0 ; u=1
            [V(k,i),idx] = min([V(k+1,i); 0.25*c(69+k)*p(i+1) + ...
                                V(k+1,i+1)]);

            % Save minimizing control action
            u(k,i) = idx-1;

        end

    end

end
Optimal Schedule

Total cost = 22.625 cents
Ex 3: Smart Appliance Scheduling w/ Random Cost

<table>
<thead>
<tr>
<th>Time of Day</th>
<th>Electricity Cost [cents/kWh]</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:00</td>
<td>0</td>
</tr>
<tr>
<td>04:00</td>
<td>5</td>
</tr>
<tr>
<td>08:00</td>
<td>10</td>
</tr>
<tr>
<td>12:00</td>
<td>15</td>
</tr>
<tr>
<td>16:00</td>
<td>20</td>
</tr>
<tr>
<td>20:00</td>
<td>25</td>
</tr>
<tr>
<td>24:00</td>
<td>30</td>
</tr>
</tbody>
</table>

Forecasted Price
Real Price

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Prof. Moura | UC Berkeley
Random Cost

True cost \[=\text{forecasted cost} + \text{random perturbation}\]
\[= c_k + w_k\]

- Variable \(w_k\) is random.
- Example probability distributions: uniform, normal, log-normal, Poisson, chi-squared, gamma, Pareto, non-parametric.
- Suppose the most basic statistic is known: the expected value.
- Let \(\overline{w}_k = E[w_k]\)
- Random cost-per-time-step: \(c_k(x_k, u_k, w_k) = \frac{1}{4}(c_k + w_k)p_{x_{k+1}}u_k\)
Stochastic Optimization

\[ \min \quad J = \mathbb{E} \left[ \sum_{k=0}^{N-1} c(x_k, u_k, w_k) + c_N(x_N) \right] \]

s. to
\[ x_{k+1} = x_k + u_k \]
\[ x_0 = 0 \]
\[ u_k \in \{0, 1\} \]
**Stochastic Dynamic Programming (SDP)**

Let $V_k(x_k)$ represent expected min cost-to-go from time step $k$ to $N$, given current state $x_k$.

**Principle of Optimality:**

$$
V_k(x_k) = \min_{u_k} E \left\{ c(x_k, u_k, w_k) + V_{k+1}(x_{k+1}) \right\}
$$

$$
= \min_{u_k \in \{0,1\}} \left\{ E \left[ \frac{1}{4}(c_k + w_k) x_{k+1} u_k \right] + V_{k+1}(x_{k+1}) \right\}
$$

$$
= \min_{u_k \in \{0,1\}} \left\{ \frac{1}{4}(c_k + \bar{w}_k) x_{k+1} u_k + V_{k+1}(x_k + u_k) \right\}
$$

$$
= \min \left\{ V_{k+1}(x_k), \frac{1}{4}(c_k + \bar{w}_k) x_{k+1} + V_{k+1}(x_k + 1) \right\}
$$

with the boundary condition

$$
V_N(5) = 0, \quad V_N(i) = \infty \text{ for } i \neq 5
$$

**Optimal Control Action:**

$$
u_k^*(x_k) = \arg\min_{u_k \in \{0,1\}} E \left\{ \frac{1}{4}(c_k + w_k) x_{k+1} u_k + V_{k+1}(x_{k+1}) \right\}
$$
Incorporate **uncertainty** as **random variables**

Need to know **some statistics** about the random variables

Minimize **expected** cost
